Problem 3.3

Show that if $\langle h \,|\, \hat{Q}h \rangle = \langle \hat{Q}h \,|\, h \rangle$ for all h (in Hilbert space), then $\langle f \,|\, \hat{Q}g \rangle = \langle \hat{Q}f \,|\, g \rangle$ for all f and g (i.e. the two definitions of "hermitian"—Equations 3.16 and 3.17—are equivalent). *Hint:* First let $h = f + g$, and then let $h = f + ig$.

Solution

Suppose that $\langle h \,|\, \hat{Q}h \rangle = \langle \hat{Q}h \,|\, h \rangle$ for all h. If $h = f + g$, then

$$\begin{aligned}
\langle h \,|\, \hat{Q}h \rangle &= \langle h \,|\, \hat{Q} \,|\, h \rangle \\
&= \int_a^b h^*(x)\hat{Q}h(x)\,dx \\
&= \int_a^b h^*(x)[\hat{Q}h(x)]\,dx \\
&= \int_a^b [f(x)+g(x)]^*\{\hat{Q}[f(x)+g(x)]\}\,dx \\
&= \int_a^b [f^*(x)+g^*(x)][\hat{Q}f(x)+\hat{Q}g(x)]\,dx \\
&= \int_a^b [f^*(x)\hat{Q}f(x)+f^*(x)\hat{Q}g(x)+g^*(x)\hat{Q}f(x)+g^*(x)\hat{Q}g(x)]\,dx \\
&= \int_a^b f^*(x)\hat{Q}f(x)\,dx + \int_a^b f^*(x)\hat{Q}g(x)\,dx + \int_a^b g^*(x)\hat{Q}f(x)\,dx + \int_a^b g^*(x)\hat{Q}g(x)\,dx \\
&= \langle f \,|\, \hat{Q} \,|\, f \rangle + \langle f \,|\, \hat{Q} \,|\, g \rangle + \langle g \,|\, \hat{Q} \,|\, f \rangle + \langle g \,|\, \hat{Q} \,|\, g \rangle \\
&= \langle f \,|\, \hat{Q}f \rangle + \langle f \,|\, \hat{Q}g \rangle + \langle g \,|\, \hat{Q}f \rangle + \langle g \,|\, \hat{Q}g \rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle \hat{Q}h \,|\, h \rangle &= \langle h \,|\, \hat{Q}^\dagger \,|\, h \rangle \\
&= \int_a^b h^*(x)\hat{Q}^\dagger h(x)\,dx \\
&= \int_a^b h^*(x)[\hat{Q}^\dagger h(x)]\,dx \\
&= \int_a^b [f(x)+g(x)]^*\{\hat{Q}^\dagger[f(x)+g(x)]\}\,dx \\
&= \int_a^b [f^*(x)+g^*(x)][\hat{Q}^\dagger f(x)+\hat{Q}^\dagger g(x)]\,dx.
\end{aligned}$$

Expand the integrand.

$$\begin{aligned}\langle \hat{Q}h \,|\, h\rangle &= \int_a^b [f^*(x)\hat{Q}^\dagger f(x) + f^*(x)\hat{Q}^\dagger g(x) + g^*(x)\hat{Q}^\dagger f(x) + g^*(x)\hat{Q}^\dagger g(x)]\,dx \\ &= \int_a^b f^*(x)\hat{Q}^\dagger f(x)\,dx + \int_a^b f^*(x)\hat{Q}^\dagger g(x)\,dx + \int_a^b g^*(x)\hat{Q}^\dagger f(x)\,dx + \int_a^b g^*(x)\hat{Q}^\dagger g(x)\,dx \\ &= \langle f \,|\, \hat{Q}^\dagger \,|\, f\rangle + \langle f \,|\, \hat{Q}^\dagger \,|\, g\rangle + \langle g \,|\, \hat{Q}^\dagger \,|\, f\rangle + \langle g \,|\, \hat{Q}^\dagger \,|\, g\rangle \\ &= \langle \hat{Q}f \,|\, f\rangle + \langle \hat{Q}f \,|\, g\rangle + \langle \hat{Q}g \,|\, f\rangle + \langle \hat{Q}g \,|\, g\rangle\end{aligned}$$

Since $\langle h \,|\, \hat{Q}h\rangle = \langle \hat{Q}h \,|\, h\rangle$ and $\langle f \,|\, \hat{Q}f\rangle = \langle \hat{Q}f \,|\, f\rangle$ and $\langle g \,|\, \hat{Q}g\rangle = \langle \hat{Q}g \,|\, g\rangle$,

$$\cancel{\langle f \,|\, \hat{Q}f\rangle} + \langle f \,|\, \hat{Q}g\rangle + \langle g \,|\, \hat{Q}f\rangle + \cancel{\langle g \,|\, \hat{Q}g\rangle} = \cancel{\langle \hat{Q}f \,|\, f\rangle} + \langle \hat{Q}f \,|\, g\rangle + \langle \hat{Q}g \,|\, f\rangle + \cancel{\langle \hat{Q}g \,|\, g\rangle}$$

$$\langle f \,|\, \hat{Q}g\rangle + \langle g \,|\, \hat{Q}f\rangle = \langle \hat{Q}f \,|\, g\rangle + \langle \hat{Q}g \,|\, f\rangle. \tag{1}$$

If $h = f + ig$, then

$$\begin{aligned}\langle h \,|\, \hat{Q}h\rangle &= \langle h \,|\, \hat{Q} \,|\, h\rangle \\ &= \int_a^b h^*(x)\hat{Q}h(x)\,dx \\ &= \int_a^b h^*(x)[\hat{Q}h(x)]\,dx \\ &= \int_a^b [f(x) + ig(x)]^*\{\hat{Q}[f(x) + ig(x)]\}\,dx \\ &= \int_a^b [f^*(x) - ig^*(x)][\hat{Q}f(x) + i\hat{Q}g(x)]\,dx \\ &= \int_a^b [f^*(x)\hat{Q}f(x) + if^*(x)\hat{Q}g(x) - ig^*(x)\hat{Q}f(x) + g^*(x)\hat{Q}g(x)]\,dx \\ &= \int_a^b f^*(x)\hat{Q}f(x)\,dx + i\int_a^b f^*(x)\hat{Q}g(x)\,dx - i\int_a^b g^*(x)\hat{Q}f(x)\,dx + \int_a^b g^*(x)\hat{Q}g(x)\,dx \\ &= \langle f \,|\, \hat{Q} \,|\, f\rangle + i\langle f \,|\, \hat{Q} \,|\, g\rangle - i\langle g \,|\, \hat{Q} \,|\, f\rangle + \langle g \,|\, \hat{Q} \,|\, g\rangle \\ &= \langle f \,|\, \hat{Q}f\rangle + i\langle f \,|\, \hat{Q}g\rangle - i\langle g \,|\, \hat{Q}f\rangle + \langle g \,|\, \hat{Q}g\rangle\end{aligned}$$

and

$$\begin{aligned}
\langle \hat{Q}h \,|\, h\rangle &= \langle h \,|\, \hat{Q}^\dagger \,|\, h\rangle \\
&= \int_a^b h^*(x)\hat{Q}^\dagger h(x)\,dx \\
&= \int_a^b h^*(x)[\hat{Q}^\dagger h(x)]\,dx \\
&= \int_a^b [f(x)+ig(x)]^*\{\hat{Q}^\dagger[f(x)+ig(x)]\}\,dx \\
&= \int_a^b [f^*(x)-ig^*(x)][\hat{Q}^\dagger f(x)+i\hat{Q}^\dagger g(x)]\,dx \\
&= \int_a^b [f^*(x)\hat{Q}^\dagger f(x)+if^*(x)\hat{Q}^\dagger g(x)-ig^*(x)\hat{Q}^\dagger f(x)+g^*(x)\hat{Q}^\dagger g(x)]\,dx \\
&= \int_a^b f^*(x)\hat{Q}^\dagger f(x)\,dx + i\int_a^b f^*(x)\hat{Q}^\dagger g(x)\,dx - i\int_a^b g^*(x)\hat{Q}^\dagger f(x)\,dx + \int_a^b g^*(x)\hat{Q}^\dagger g(x)\,dx \\
&= \langle f \,|\, \hat{Q}^\dagger \,|\, f\rangle + i\langle f \,|\, \hat{Q}^\dagger \,|\, g\rangle - i\langle g \,|\, \hat{Q}^\dagger \,|\, f\rangle + \langle g \,|\, \hat{Q}^\dagger \,|\, g\rangle \\
&= \langle \hat{Q}f \,|\, f\rangle + i\langle \hat{Q}f \,|\, g\rangle - i\langle \hat{Q}g \,|\, f\rangle + \langle \hat{Q}g \,|\, g\rangle.
\end{aligned}$$

Since $\langle h \,|\, \hat{Q}h\rangle = \langle \hat{Q}h \,|\, h\rangle$ and $\langle f \,|\, \hat{Q}f\rangle = \langle \hat{Q}f \,|\, f\rangle$ and $\langle g \,|\, \hat{Q}g\rangle = \langle \hat{Q}g \,|\, g\rangle$,

$$\begin{aligned}
\cancel{\langle f \,|\, \hat{Q}f\rangle} + i\langle f \,|\, \hat{Q}g\rangle - i\langle g \,|\, \hat{Q}f\rangle + \cancel{\langle g \,|\, \hat{Q}g\rangle} &= \cancel{\langle \hat{Q}f \,|\, f\rangle} + i\langle \hat{Q}f \,|\, g\rangle - i\langle \hat{Q}g \,|\, f\rangle + \cancel{\langle \hat{Q}g \,|\, g\rangle} \\
i\langle f \,|\, \hat{Q}g\rangle - i\langle g \,|\, \hat{Q}f\rangle &= i\langle \hat{Q}f \,|\, g\rangle - i\langle \hat{Q}g \,|\, f\rangle \\
\langle f \,|\, \hat{Q}g\rangle - \langle g \,|\, \hat{Q}f\rangle &= \langle \hat{Q}f \,|\, g\rangle - \langle \hat{Q}g \,|\, f\rangle. \qquad (2)
\end{aligned}$$

Add the respective sides of equations (1) and (2).

$$2\langle f \,|\, \hat{Q}g\rangle = 2\langle \hat{Q}f \,|\, g\rangle$$

Therefore, dividing both sides by 2,

$$\langle f \,|\, \hat{Q}g\rangle = \langle \hat{Q}f \,|\, g\rangle$$

for all f and g.